Relaxation properties of small-world networks

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Recently, Watts and Strogatz introduced the so-called small-world networks in order to describe systems that combine simultaneously properties of regular and random lattices. In this work we study diffusion processes defined on such structures by considering explicitly the probability for a random walker to be present at the origin. The results are intermediate between the corresponding ones for fractals and Cayley trees.

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I. INTRODUCTION

Networks of the real world often seem to combine aspects from regular and completely random lattices. Social networks, neural networks, electrical power grids, and traffic networks [1–3] are all examples of patterns not described satisfactorily by conventional regular lattices, nor by completely random lattices. Social structures, for instance, do not behave as regular lattices, since (as is well known) randomly chosen people are connected in general by a small number of intermediary bilateral ties. Here, as in random graphs, the minimal (chemical) distance between any two points in the system scales logarithmically with the system size [4].

To combine these two properties, Watts and Strogatz recently introduced the idea of small-world networks [1]. This construction is a superposition of a regular lattice on a random lattice, and includes simultaneously well defined local clusters and short global connections. As we will demonstrate, these systems also display properties intermediate between those of regular and treelike (loopless) lattices, even under a small number of global connections, provided the system size is large enough.

Much work has already been done on the properties of small-world networks [1,2,5–14] but most of it has focused on static (geometric) properties. We shall not address these issues, but rather concentrate on a *dynamical* model defined on the structure. Treatments of the dynamics of small-world networks include, for instance, the study of an Ising model defined on the lattice [5], spectral properties of the small-world Laplacian [6], percolation [7], spreading of diseases [8], and neural networks [9]. In the following we will examine the properties of random walks on small-world networks, in particular, the relaxation, exemplified by the probability for a random walker to be at the original site at a later time. This is a simple quantity to extract numerically, and very relevant for various physical properties: It is sensitive to the topology of the network and is related to its vibrational modes.

II. DEFINITION OF THE MODEL AND PRESENTATION OF THE RESULTS

The small-world networks we consider are built as follows: We start from a regular lattice with L vertices in one dimension under periodic boundary conditions, each site being connected symmetrically to its $2k$ nearest neighbors, i.e., having as coordination number $z=2k$. Then we add to each of the sites a new bond with probability p. The other end gets attached with equal probability to any of the lattice sites; this also allows the possibility of vertices to become connected to themselves. In this way we add, independent of k, on the average pL new bonds to the underlying regular lattice. This construction follows [7] for $k=1$ and is simpler than the original procedure [1], by which one rewires each of the original kL bonds randomly with probability p.

A stepwise diffusion process is now defined by specifying all the transition probabilities $W_{i,j}$ entering the master equation:

$$P(i,n+1)-P(i,n)=\sum_j W_{i,j}P(j,n)-P(i,n)\sum_j W_{j,i}. \tag{1}$$

The $W_{f,i}$ is the probability of going from site i to site f during one time step, and the probability $P(i,n)$, $i=1,\ldots,L$ is just the probability of being at site i after the nth step. The process defined in Eq. (1) is the discrete-time variant of diffusion on an arbitrary lattice, a topic interesting in its own right. Diffusion on regular lattices is ubiquitous, and diffusion on random graphs has (among other things) also been studied in the context of glassy relaxation [15]. We are therefore inspired to investigate what happens on the small-world model, which interpolates between these two extremes. Previously a lot of interest has also been seen in the related problem of diffusion on fractals (see, for example, [16–19] and references therein). As we proceed to show, diffusion on Cayley trees [20–22] also shows features closely related to the present problem. Furthermore, the motion of charge carriers or of excitons over polymer chains, where steps between spatially close sites can connect regions far apart along the chemical backbone, also involves global shortcuts [23,24].

The transition probabilities $W_{i,j}$ in Eq. (1) are as follows. First $W_{i,j}=0$ if there are no bonds between i and j. For i connected to j by one or more direct bonds, $W_{i,j}$ is proportional to the number of such bonds. The same holds for the probability of remaining at the same site after one time unit, i.e., we allow ''sticking.'' Formally

$$W_{i,j}=\frac{z_{i,j}+\delta_{i,j}}{z_j+1}. \tag{2}$$

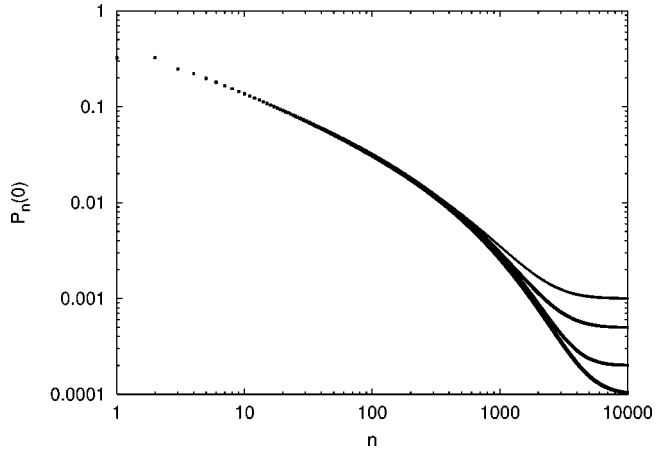

FIG. 1. The relaxation or probability of presence at the origin $P_n(0)$ as a function of number of steps for $p=0.05$ and several system sizes L, which from upper to lower right are $L=1000$, $L=2000$, $L=5000$, and $L=10\,000$.

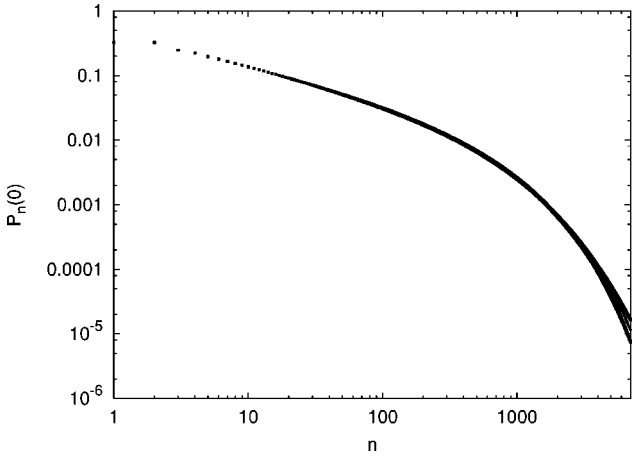

FIG. 2. Plot of $P_n(0)-P_\infty(0)$ as a function of n, the number of time steps for $p=0.05$ and L as in Fig. 1. The curves fall nicely on a master curve.

In this equation, $z_{i,j}$ is the number of bonds between the two sites i and j, and z_i is the total number of bonds emanating from vertex i, i.e., the coordination number of the site. Hence $z_i=\Sigma_j z_{i,j}$. Note that the $z_{i,j}$ values are determined by the additional wiring as well as by the underlying lattice. The $\delta_{i,j}$ and the 1 in the denominator appear because we allow for the possibility of the walker to remain at site i during a time step. This procedure renders the numerically determined $P(i,n)$ smoother in n. We remark that the rates defined according to Eq. (2) are not symmetrical in i and j, i.e., in general $W_{i,j}\neq W_{j,i}$.

The algorithm we have used is the exact (cellular automaton) enumeration of random walks [18], corresponding to the implementation of Eq. (1). All the results plotted are averaged over 200 disorder configurations. We have worked mostly with the value $k=1$. This is also the value implied if we do not state otherwise.

We focus on the probability $P(i,n|i,0)$ that a particle initially at site i is found at the same site just after the nth step. In the figures below we plot $\langle P(i,n|i,0)\rangle$, i.e., the average of $P(i,n|i,0)$ over the different realizations of the small-world lattice. Since all sites are equivalent in an ensemble of small-world networks, this quantity does not depend on the particular site i chosen, and we hereafter denote it by $P_n(0)$. In Fig. 1 we have chosen $p=0.05$ and plotted $P_n(0)$ on a double-logarithmic scale for system sizes ranging from $L=1000$ to $L=10\,000$. This allows us to examine the dependence of $P_n(0)$ on the size of the system.

From Fig. 1 we infer that initially all the curves fall on one curve and that for large n they saturate at their respective equilibrium values $1/L$. However, Eq. (2) implies an inhomogeneous equilibrium distribution

$$P^{eq}(i)\propto(z_i+1). \tag{3}$$

Therefore $P(i,\infty|i,0)$ depends on the specific small-world realization, and will fluctuate from realization to realization around its average value $1/L$.

To find out how much of the behavior is due to finite size effects, we subtract from each average curve in Fig. 1 its corresponding average equilibrium value $P_\infty(0)\equiv 1/L$, and replot $P_n(0)-P_\infty(0)$ in Fig. 2. From Fig. 2 we see that all curves collapse nicely onto what we view as representing $P_n(0)$ on small-world networks in the limit $L\to\infty$. Both Figs. 1 and 2 display initially a quasilinear decay on the chosen double-logarithmic scale, and this may be viewed as being an approximate power-law decay. Depending on p, the exponents range from around -0.5 for the smallest p to around -0.6 for the largest. This regime is followed by a steeper decay at larger n. To highlight the power-law character we have plotted in Fig. 3 $P_n(0)$ for $p=0.01$. As is evident from the figure, the power-law domain extends well over two orders of magnitude in n.

The results can be understood qualitatively in the following way. For a fractal one has [25]

$$P_n(0)\sim n^{-d_s/2}, \tag{4}$$

where d_s is the spectral dimension. Thus the initial decay in Figs. 1–3 follows that of a fractal with a d_s close to 1, i.e., that of a quasi-one-dimensional (quasi-1D) system. This is reasonable given our construction: for sufficiently small p and small n, only relatively few random walkers encounter any long-range connections (shortcuts). Therefore in the be-

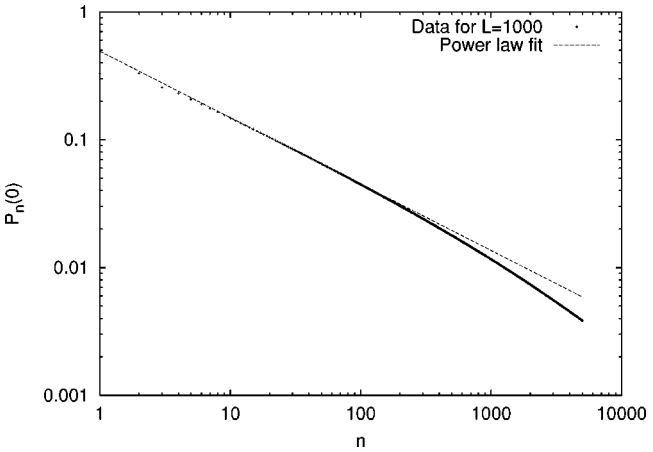

FIG. 3. $P_n(0)$ for $p=0.01$ as well as a power-law approximation. The fit in the region $10<t<100$ gives as least-squares-fit exponent -0.52.

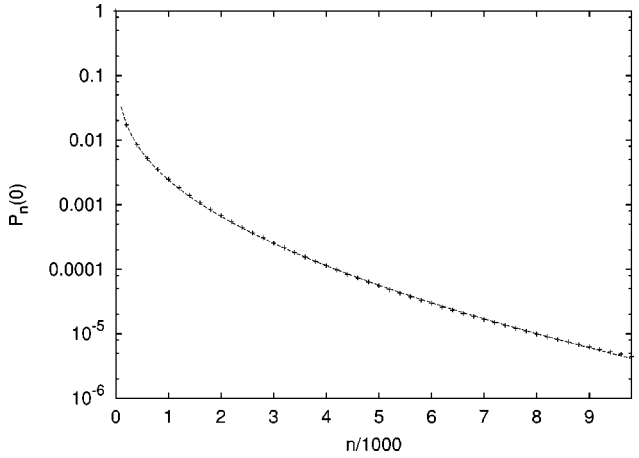

FIG. 4. $P_n(0)-P_\infty(0)$ for $p=0.05$ and $L=10\,000$ on a semilogarithmic scale. At longer times the decay appears to be slower than exponential. Also shown is a fit to a stretched exponential, indistinguishable from the data.

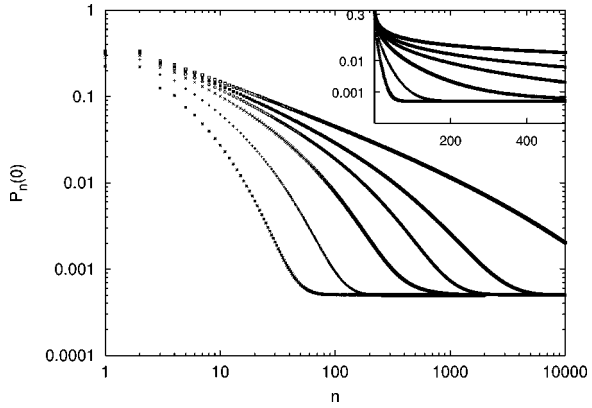

FIG. 5. The relaxation or probability of presence at the origin as a function of the number of steps for $L=2000$ and from upper right to lower left $p=0.01$, 0.05, 0.1, 0.2, 0.4, and 0.8. The data for $0\leqslant n\leqslant 500$ are replotted semilogarithmically in the inset.

ginning the behavior of $P_n(0)$ closely reflects the character of the underlying 1D lattice. However for larger n, the random walkers probe larger and larger portions of the graph, and thus follow more and more shortcuts. This progressively speeds up the decay of $P_n(0)$ as more regions at larger and larger length scales are visited, and the fractal picture is lost. One would thus expect that the concept of a d_s begins to be invalid when the random walkers visit enough shortcuts, i.e., when the 1D diffusion extends farther than the typical distance between shortcuts. This is the fundamental length scale ξ of small-world networks, in addition to the lattice constant, which is less important here. In our case we have

$$\xi = p^{-1}, \tag{5}$$

measuring ξ in units of the lattice constant. For diffusion on scales smaller than ξ one furthermore has in terms of the diffusion constant D of the regular lattice $\xi^2 \sim 2Dn$, so that

$$n \sim \frac{1}{2Dp^2}. \tag{6}$$

Given that we allow random walkers to stay at a site during a time step, $D=1/3$ and thus $n=2/3p^{-2}$. However, some walkers do encounter shortcuts at length scales below $\sim p^{-1}$, and numerically the crossover to a region that does not have approximate power-law character is seen to take place earlier than $n\sim p^{-2}$.

We turn now to the analysis of this region. To be able to follow it more closely, we replot the results of Fig. 2 for $L=10\,000$ on a semilogarithmic scale in Fig. 4. Evidently, the decay for larger n is slower than exponential.

The decay of $P_n(0)$ is hence quicker than a power law, but slower than that for Cayley trees, for which one has (for coordination numbers greater than 2) [21,22]

$$P_n(0) \sim n^{-3/2} \exp(-Cn), \tag{7}$$

where C is a constant. Comparing this behavior to that displayed in Figs. 1 and 4 we remark that in our case, for a relatively small number of steps, the decay varies approximately as $n^{-\alpha}$ with $\alpha \gtrsim 1/2$, whereas at larger n a more adequate description would be a stretched exponential, $\exp(-Cn^\beta)$. One may even suspect that the decay in Fig. 4 obeys $P_n(0) \sim n^{-\alpha} \exp(-Cn^\beta)$, as also supported by the small-world network spectral density [6]. A fit of the data in Fig. 4 to this functional form (keeping $\alpha \equiv 0.5$ fixed) is also shown (with $\beta=0.56$ and $C=0.04$). We note that a similarly good fit is obtained by setting $\beta=1/3$ (in line with [6]); then $\alpha=0.1$ and $C=0.53$.

We now consider the dependence of the decay on the value of p. For this we plot in Fig. 5 the decay law $P_n(0)$ for $L=2000$ and p ranging from $p=0.01$ to $p=0.8$. To bring out the stretched exponential behavior, the data for $0\leqslant n\leqslant 500$ are also displayed semilogarithmically in the inset. We note that the initial power-law-like region diminishes with increasing p. Furthermore, the plateau region $P_n(0)\simeq 1/L$ is reached earlier for larger p. This is in accordance with our argument above, that the long-range connections (shortcuts) interrupt the simple diffusion on the underlying lattice, such that the crossover length decreases with increasing p [cf. also

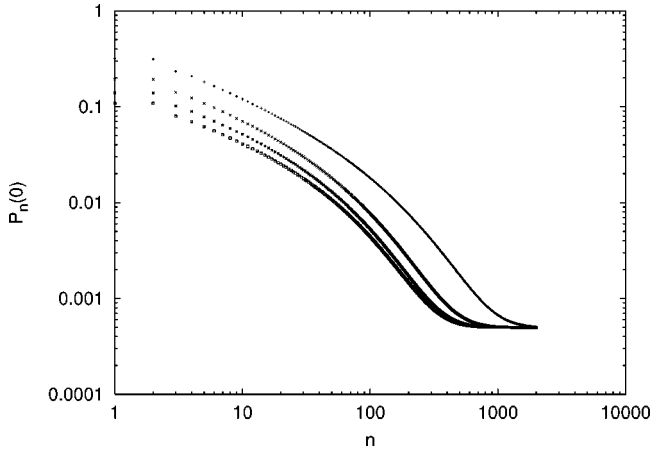

FIG. 6. $P_n(0)$ plotted for $L=2000$, $p=0.1$, and $k=1$, 2, 3, and 4, from right to left.

Eq. (6)]. As p becomes large enough, the power-law regime essentially disappears. This is so because the random walker rapidly meets a shortcut. As before, the influence of finite size effects can be reduced by plotting, as in Fig. 2, $P_n(0) - P_\infty(0)$.

We have also performed simulations of the random walk on small-world networks where the underlying lattice has a k value larger than 1. In Fig. 6 we plot the results for $p=0.1$ and $L=2000$ in the cases of $k=1$, $k=2$, $k=3$, and $k=4$. The findings reproduce the general picture: $P_n(0)$ behaves like a power law for small n, while decaying more rapidly as n gets larger. The curves for different k are mainly shifted with respect to each other, and the network with the largest coordination number (largest k) also displays the quickest relaxation. It is to be noted, however, that the case $k=1$ has the largest dynamical range and thus best shows the decay forms, while also being the one simplest to implement; hence $k=1$ may be the ideal small-world model.

III. CONCLUSIONS

In this work we have studied numerically the behavior of random walks on small-world lattices. Our work has focused on the probability of being at the initial site $P_n(0)$ as a function of the number of steps n. This quantity is found to show a complex, very interesting pattern. Initially $P_n(0)$ displays a power-law, ''quasifractal'' regime. At larger n a quicker decay takes over, reminiscent of stretched exponentials. In this respect the $P_n(0)$ decay is intermediate between the decays found for fractal structures and those found for treelike (loopless) structures, exemplified here by Cayley trees.

ACKNOWLEDGMENTS

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